Advanced State Space Grids
- definitions and proofs -

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1 Fundamental definitions

Definition 1.1: Grid of an ASSG plot

Let the ASSG method only include d indicators with a possible ranking between 1 and $o \in \mathbb{N}$. Then the plot of an ASSG is an illustration of the vector space in $[1, o]^d \subset \mathbb{R}^d$. The grid of an ASSG plot is then a vector subspace in $\{1, ..., o\}^d \subset \mathbb{N}^d$.

Definition 1.2: Ratings of a single indicator

Let x_1 be the random variable of an indicator on the x_1-axis of the ASSG plot. Then $\{x_{1,1}, x_{1,2}, x_{1,n}\}$ with $(x_{1,j})_{j \in \{1,...,n\} \subset \mathbb{N}}$ are the ratings of the indicator on the x_1-axis with the total number of n ratings. Likewise the i-th indicator which is on the x_i-axis is defined as x_i with the ratings $\{x_{i,1}, x_{i,1}, ..., x_{i,n}\}$. In this case, every rating $x_{i,j}$ is a random variable and x_i is a set of the random variables $(x_{i,j})_{i \in \{1,...,d\}, j \in \{1,...,n\}}$.

Definition 1.3: Ratings of a single indicator with multiple raters

Let x_i be an indicator with n ratings. Additionally let each rating be rated by p raters. Then each rating $x_{i,j}$ is rated p times with the ratings $\{x^1_{i,j}, x^2_{i,j}, ..., x^p_{i,j}\}$.
Each rating $x_{i,j}$ for indicator x_i is defined by

$$x_{i,j} := \left[\frac{1}{p} \sum_{k=1}^{p} x^k_{i,j} \right]. \tag{1.1}$$

The error of each rating $\Delta x_{i,j}$ results in

$$\Delta x_{i,j} := \frac{1}{\sqrt{p}} \sqrt{\frac{1}{p} \sum_{k=1}^{p} \left(x^k_{i,j} - x_{i,j}\right)^2}. \tag{1.2}$$

Additionally the average of the errors of indicator x_i is defined by

$$\overline{\Delta x_i} = \frac{1}{n} \sum_{j=1}^{n} \Delta x_{i,j}. \tag{1.3}$$

Definition 1.4: Points in an ASSG

Let x_i be the indicators of the d-dimensional ASSG method with n ratings in each indicator. Then every point in the ASSG is defined by $(x_{1,j}, x_{2,j}, ..., x_{d,j})^T$ for all $j \in \mathcal{I} = \{1, ..., n\} \subset \mathbb{N}$.

2 Expected value

Lemma 2.1: Expected value for a single indicator

Let x_i be the i-th indicator of the ASSG method with a total number of ratings n. Then the expected value of the indicator x_i is

$$\mathbb{E}(x_i) \approx \frac{1}{n}\sum_{j=1}^{n} x_{i,j}. \tag{2.1}$$

Proof 2.1: Expected value for a single indicator

For this proof the corollary of the weak law of large numbers (Klenke 2020) can be used.To use this corollary, one must show that the indicator x_i satisfies the weak law of large numbers.

The set of random variables $(x_{i,j})_{j\in\mathbb{N}}$ are the outcomes of each time point of the indicator x_i. This means that the set of random variables $(x_{i,j})_{j\in\{1,\ldots,n\}}$ is the set that must satisfy the requirements of the weak law of large numbers.

The first premise is that the random variables $x_{i,j}$ belong to the same distribution. The rating of each indicator has been interpreted as a random variable and an unknown probability measure. Therefore, the set of the same random variables $x_{i,j}$ have the same distribution.

The next condition is that $Var(|x_i|) < \infty$. Since the results of each $x_{i,j}$ are between 1 and the highest ranking of the indicator $o \in \mathbb{N}$, the results are limited. Therefore, the highest possible variance would be $\mathbb{E}((x_i)^2)$ with $\mathbb{E}(x_i) = 0$. The highest possible value of $\mathbb{E}(x_i)^2$ is when the expected value is equal to the highest outcome. Therefore, the highest possible variance would be o^2, which is less than infinity. Thus, $Var(|x_i|) < \infty$ is satisfied.

As an approximation, it is assumed that the random variables are independent. The last premise of the weak law of large numbers is fulfilled if the set of random variables is independent (Klenke 2020).

Thus all conditions are fulfilled and the weak law of large numbers can be used. This then proves eq. 2.1 which was to be shown. □

Definition 2.1: Expected value for ASSGs

Let $(x_i)_{i\in\{1,\ldots,d\}}$ be the indicators of the ASSG method. Then the EV for the ASSG plot of these two indicators is

$$EV = \left(\mathbb{E}(x_1), \ldots, \mathbb{E}(x_d)\right)^T. \tag{2.2}$$

Lemma 2.2: Standard deviation for a single indicator

Let (x_i) be the i-th indicators of the ASSG method. Then the standard deviation $\sigma(x_i)$ of this indicator is

$$\sigma(x_i) \approx \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}(x_i) - x_{i,j}\right)^2}. \tag{2.3}$$

3 Deviation ellipseoid

Proof 3.1: Standard deviation for a single indicator

Let x_i be the i-th indicator of the ASSG method. With the definition of the standard deviation and the variance , the standard deviation can be formulated into combinations of expected values. Corollary 2.1 can then be used to approximate the expected value. This leads to following proof:

$$\begin{aligned}\sigma(x_i) &= \sqrt{Var(x_i)}\\ &= \sqrt{\mathbb{E}\left((\mathbb{E}(x_i) - x_{i,j})^2\right)}\\ &\approx \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mathbb{E}(x_i) - x_{i,j})^2}.\end{aligned}$$

□

Definition 3.1: Mean standard deviation

Let $(x_i)_{i\in\{1,\dots,d\}}$ be the indicators of the ASSG method. Then the MSD of these indicators is

$$MSD = \frac{1}{d}\sum_{i=1}^{d}\sigma(x_i). \tag{3.1}$$

Definition 3.2: Deviation ellipse

Let x_i and x_k be two indicators of the ASSG method. Then the deviation ellipse of these indicators describes the following closed curve with the form of an ellipse

$$DE(\phi) = \begin{pmatrix}\mathbb{E}(x_i) + \sigma(x_i)cos(\phi)\\ \mathbb{E}(x_k) + \sigma(x_k)sin(\phi)\end{pmatrix}, \text{with } \phi\epsilon[0, 2\pi). \tag{3.2}$$

Definition 3.3: Deviation ellipsoid

Let x_i, x_k and x_l be three indicators of the ASSG method. Then the deviation ellipsoid of these indicators describes the following area with the form of an ellipsoid:

$$DE(\phi, \theta) = \begin{pmatrix}\mathbb{E}(x_i) + \sigma(x_i)sin(\phi)cos(\theta)\\ \mathbb{E}(x_k) + \sigma(x_k)sin(\phi)sin(\theta)\\ \mathbb{E}(x_l) + \sigma(x_l)cos(\theta)\end{pmatrix}, \text{with } \phi \in [0, 2\pi) \text{ and } \theta \in [0, \pi). \tag{3.3}$$

4 Standard deviation difference

Definition 4.1: Standard deviation difference for d indicators

Let $(x_i)_{i\in\{1,\ldots,d\}}$ be the indicators of the ASSG method. Then the standard deviation difference is

$$SDD = \frac{2}{d(d-1)} \sum_{i=1}^{d} \sum_{k=1}^{i} |\sigma(x_i) - \sigma(x_k)|. \tag{4.1}$$

5 Chi-squared Tests

Definition 5.1: χ^2 test for stationary fits with fixed x_i value

Let c_i be a constant for the stationary $\chi^2_{x_i}$ test with fixed x_i value. Then the result of the χ^2_x test is

$$\chi^2_{x_i} = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_{i,j} - c_i}{x_{i,j} - \mathbb{E}(x_i)} \right)^2. \tag{5.1}$$

Definition 5.2: χ^2 test for stationary fits with fixed x_i value with multiple raters

Let c_i be a constant for the stationary $\chi^2_{x_i}$ test with fixed x_i value with errors $\Delta x_{i,j}$. Then the result of the χ^2_x test is

$$\chi^2_{x_i} = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_{i,j} - c_i}{\Delta x_{i,j}} \right)^2. \tag{5.2}$$

In case $\Delta x_{i,j} = 0$ let

$$\Delta x_{i,j} = \frac{\overline{\Delta x_i}}{n} \tag{5.3}$$

be instead.

Definition 5.3: χ^2 test for linear fits

Let f be a linear function for the χ^2 test. Also let x_i and x_k be two indicators of the ASSG method each with a total number of n ratings. Then the result of the χ^2 test is

$$\chi^2_{x_i,x_k} = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_{k,j} - f(x_{i,j})}{x_{i,j} - \mathbb{E}(x_i)} \right)^2. \tag{5.4}$$

Definition 5.4: χ^2 test for fits with multiple raters

Let $f : [0,5] \to [0,5]$, $t \to f(t)$ be a function for the χ^2 test of indicator x_i and x_k with $m, b \in \mathbb{R}$. Also let x_i and x_k be indicators of the ASSG method each with a total number of n ratings. Additionally let $\Delta x_{i,j}$ and $\Delta x_{k,j}$ be the errors of indicator x_i and x_j. Then the result of the χ^2 test is

$$\chi^2_{x_i,x_k} = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(min(\left|(x_{i,j}, x_{k,j})^T - (t, f(t))^T\right|)\right)^2}{\sqrt{\Delta x_{i,j}^2 + \Delta x_{k,j}^2}}. \tag{5.5}$$

In case $\Delta x_{i,j} = \Delta x_{k,j} = 0$ let

$$\Delta x_{i,j} = \frac{\overline{\Delta x_i}}{n} \text{ and } \Delta x_{k,j} = \frac{\overline{\Delta x_k}}{n} \tag{5.6}$$

be instead.

Lemma 5.1: χ^2 test for linear fits with multiple raters

Let $f : [0,5] \to [0,5]$, $t \to mt + b$ be a linear function for the χ^2 test of indicator x_i and x_k with $m, b \in \mathbb{R}$. Also let x_i and x_k be indicators of the ASSG method each with a total number of n ratings. Additionally let $\Delta x_{i,j}$ and $\Delta x_{k,j}$ be the errors of indicator x_i and x_j. Then the result of the χ^2 test is

$$\chi^2_{x_i,x_k} = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\sqrt{\Delta x_{i,j}^2 + \Delta x_{k,j}^2}} \sqrt{\left(x_{i,j} - \frac{x_{i,j} + m x_{k,j} + mb}{m^2+1}\right)^2 + \left(x_{k,j} - m\left(\frac{x_{i,j} + m x_{k,j} + mb}{m^2+1}\right) - b\right)^2}. \tag{5.7}$$

Proof 5.1: χ^2 test for linear fits with multiple raters

Use definitions of the lemma above.

$$\begin{aligned}
\left|(x_{i,j},x_{k,j})^T-(t,f(t))^T\right| &= \sqrt{(x_{i,j}-t)^2+(x_{k,j}-f(t))^2}\\
\Leftrightarrow \left(\left((x_{i,j},x_{k,j})^T-(t,f(t))^T\right)\right)^2 &= (x_{i,j}-t)^2+(x_{k,j}-f(t))^2\\
&= (x_{i,j}-t)^2+(x_{k,j}-(mt+b))^2\\
&= x_{i,j}^2-2x_{i,j}t+t^2+x_{k,j}^2-2x_{k,j}(mt+b)+(mt+b)^2\\
&= t^2+m^2t^2-2x_{i,j}t-2mx_{k,j}t+2mbt+x_{i,j}^2+x_{k,j}^2-2x_{k,j}b+b^2\\
&= t^2(m^2+1)+t(-2x_{i,j}-2mx_{k,j}+2mb)+x_{i,j}^2+x_{k,j}^2-2x_{k,j}b+b^2
\end{aligned}$$

Next step:

$$\begin{aligned}
\frac{d}{dt}\left(\left|(x_{i,j},x_{k,j})^T-(t,f(t))^T\right|\right)^2 &= 0\\
\Leftrightarrow 2t(m^2+1)-2x_{i,j}-2mx_{k,j}+2mb &= 0\\
\Leftrightarrow t &= \frac{x_{i,j}+mx_{k,j}+mb}{m^2+1}
\end{aligned}$$

$(t,f(t))^T$ is global minimum, because:

$$\frac{d^2}{dt^2}\left(\left|(x_{i,j},x_{k,j})^T-(t,f(t))^T\right|\right)^2 = 2m^2+2>0.$$

Now calculate $min\left((x_{i,j},x_{k,j})^T-(t,f(t))^T\right)$ with this new found t:

$$\begin{aligned}
min\left(\left|(x_{i,j},x_{k,j})^T-(t,f(t))^T\right|\right) &= \sqrt{(x_{i,j}-t)^2+(x_{k,j}-f(t))^2}\\
&= \sqrt{\left(x_{i,j}-\frac{x_{i,j}+mx_{k,j}+mb}{m^2+1}\right)^2+\left(x_{k,j}-m\left(\frac{x_{i,j}+mx_{k,j}+mb}{m^2+1}\right)-b\right)^2}.
\end{aligned}$$

□

6 Travel distance

Definition 6.1: Travel distance for d indicators

Let $(x_i)_{i\in\{1,\ldots,d\}}$ be the indicators of the ASSG method each with a total number of n ratings. Then the travel distance of the ASSG method is

$$td = \frac{1}{n-1}\sum_{j=1}^{n-1}\left|(x_{1,j},\ldots x_{d,j})^T-(x_{1,j+1},\ldots,x_{d,j+1})^T\right|. \tag{6.1}$$

Lemma 6.1: Travel distance of a single indicator

Let x_i be an indicator of the ASSG method each with a total number of n ratings. Then the travel distance of this indicator x_i is

$$td_{x_i} = \frac{1}{n-1}\sum_{j=1}^{n-1}\left|x_{i,j}-x_{i,j+1}\right|. \tag{6.2}$$

Proof 6.1: Travel distance of a single indicator

Let $(x_i)_{i\in\{1,...,d\}}$ be the indicators of the ASSG method each with a total number of n ratings. In case one wants to define the travel distance of a single indicator $d = 1$ holds. This leads to:

$$\begin{aligned} td_{x_i} &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|(x_{1,j}, ...x_{d,j})^T - (x_{1,j+1}, ..., x_{d,j+1})^T\right| \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|(x_{1,j}, ...x_{1,j})^T - (x_{1,j+1}, ..., x_{1,j+1})^T\right| \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|x_{1,j} - x_{1,j+1}\right| \end{aligned}$$

Since there is no order in the numbering of the indicators following holds without limitation of generality:

$$\begin{aligned} td_{x_i} &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|x_{1,j} - x_{1,j+1}\right| \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|x_{i,j} - x_{i,j+1}\right| \end{aligned}$$

□

7 Inner point density

Definition 7.1: Inner point density

Let $(x_i)_{i\in\{1,...,d\}}$ be the indicators of the ASSG method each with a total number of n ratings. Then the inner point density of the ASSG method is

$$ipd = \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\left\{j\in\{1,...,n\}|MSD>|EV-(x_{1,j},...,x_{d,j})^T|\right\}}(i). \tag{7.1}$$

8 Travel tendency

Definition 8.1: Travel tendency for d indicators

Let $(x_i)_{i\in\{1,...,d\}}$ be the indicators of the ASSG method each with a total number of n ratings. Then the travel tendency of the ASSG method is

$$tt = \frac{1}{d(n-1)}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left(\mathbb{1}_{\{j\in\mathcal{J}|x_{i,j+1}-x_{i,j}>0\}}(j) - \mathbb{1}_{\{j\in\mathcal{J}|x_{i,j+1}-x_{i,j}<0\}}(j)\right) \tag{8.1}$$

with $\mathcal{J} = \{1, ..., n-1\}$ and $\mathcal{I} = \{1, ..., d\}$.

Lemma 8.1: Travel tendency of a single indicator

Let x_i be an indicator of the ASSG method each with a total number of n ratings. Then the travel tendency of this indicator x_i is

$$tt_{x_i} = \frac{1}{n-1} \sum_{j \in \mathcal{J}} \left(\mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) - \mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} < 0\}}(j) \right) \tag{8.2}$$

with $\mathcal{J} = \{1, ..., n-1\}$.

Proof 8.1: Travel tendency of a single indicator

Let $(x_i)_{i \in \{1,...,d\}}$ be the indicators of the ASSG method each with a total number of n ratings. Additonally let:

$$\mathcal{J} = \{1, ..., n-1\} \text{ and } \mathcal{I} = \{1, ..., d\}$$

In case one wants to define the travel distance of a single indicator $d = 1$ holds. This leads to:

$$\mathcal{I} = \{1, ..., d\} = \{1\}$$

$$\begin{aligned} tt &= \frac{1}{d(n-1)} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left(\mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) - \mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) \right) \\ &= \frac{1}{1 \cdot (n-1)} \sum_{i \in \{1\}} \sum_{j \in \mathcal{J}} \left(\mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) - \mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) \right) \\ &= \frac{1}{n-1} \sum_{j \in \mathcal{J}} \left(\mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) - \mathbb{1}_{\{j \in \mathcal{J} | x_{i,j+1} - x_{i,j} > 0\}}(j) \right) \end{aligned}$$

□

References

[1] Klenke, A. (2020): Wahrscheinlichkeitstheorie (4. edition). Springer Spektrum.